Approaches to the σ = ½ phenomenon in multiplicative number theory – Peter Braun

Abstract:

In this note we explain why the line of convergence of $\sum\mu(n)/n^s$ is undecidable in (½, 1) in arithmetic. The Riemann hypothesis follows as a consequence as does the simplicity of the zeros. The method extends to zeta functions which satisfy certain minimal conditions. The explanation is intuitive to the extent that we are writing about something that is not there – inductive pattern – but some attempt is made to quantify this notion.

Introduction

The argument here for the zeta phenomenon is very straight forward and does not involve disappearing into some complex mathematical, logical or philosophical mind space where the torture of such a journey makes almost anything seem reasonable. On the contrary we just need to be very clear that inductive reasoning is at the very base of sensible reasoning and some time is spent working around this topic as the method should have application to other problems in number theory. Another important part of things is the human capacity to recognise. Th**e** formula for the sum of the first N natural numbers is about recognising pattern in the step from k to k+1 in the inductive proof. This has little to do with logical reasoning but is a cognitive quality that many people enjoy. A problem in the following discussion is about trying to recognise non-pattern by putting together guidelines using inductive reasoning in the general sense. This may seem somewhat paradoxical but the attempt is made in this note to achieve this goal.

The explanation could be reduced to a few of lines once there is an understanding of the notion of non-inductive statements

Mathematicians use the familiar combination of inductive and deductive reasoning to discuss things. The inductive reasoning is the inner reasoning in which not everything is assumed and it is the recognition of pattern that suggests extension in thought. Deductive reasoning is the outer reasoning which has all the pieces in the argument defined and understood. But the two forms of reasoning are used in tandem to create new entities inductively, which are then used in deductive reasoning.

Throughout this discussion arithmetic will mean Peano arithmetic without the assumption of universal quantification and complete induction. It is the arithmetic, available to anybody using inductive and deductive reasoning within normal classical logic. Complex analysis will then be seen as the argument system and language which includes complete induction and universal quantification in line with ZFC set theory.

The view adopted in this explanation is that the language of complex analysis used to discuss the theory of the Riemann zeta function may be developed from arithmetic without the assumption of universal quantification. This is in part a return to Kronecker's arithmetic where we take the view that we make up complex analysis from finite arithmetic using everyday inductive reasoning. In the more formal theory, there is the additional ***assumption*** that it is sensible to write about the entities created in the construction of the real numbers as these are **assumed** things, the existence of which, for the most part, is not verifiable. They are not the results of finite construction. We are however, able to use inductive reasoning to be convinced, using substitution and approximation, that the results about arithmetic which are derived in this wider language cannot be contradicted in arithmetic. We also know from the

results of formal logic that there are theorems in complete arithmetic which cannot be proven within that system.

The notions of function, transformation and process are all ideas which allow the concept of inverse. The overview of the discussion is to see aspects of the theory of the Riemann zeta function defining in the language of analysis, a transformation which is non-invertible. We see a move from an inductive state to an equivalent non-inductive state in the language shift from arithmetic to analysis in the theory of the zeta function. This is seen to produce statements which are non-inductive because of their multiplicative foundations and non-provable in arithmetic. Computing calculations will be seen to be an extension of the base arithmetic and this is used to argue the arithmetic completeness of the computing phenomenon associated with the Riemann hypothesis.

An additional '*assertion*' will be introduced which in other places may be viewed as a companion '*axiom*', to the axiom of infinity of formal set theory which provides arithmetic with complete induction.

If in arithmetic, an unbounded number of thought processes are necessary for proof of a collection of theorems, then there does not exist a provable theorem in arithmetic which establishes the truth of these theorems.

In line with this we see there is a strong limitation on the global link between the multiplicative numbers and the inductive numbers in arithmetic. That is, we can have an understanding of the inductive numbers finitely (n+1 is the successor of n) but we have to know more and more about multiplicative structure to have a corresponding understanding of multiplicative structure. Elementary proofs of the prime number theorem, Breusch [2] pretty well define the limitations of the local link with the ordered prime numbers. The theorem is local in the sense that it is about the primes less than or equal to x even though it is usually written as $\pi(x) = x/\ln(x) + o\,(x/\ln(x))$ as $x \to \infty$. The theory of the Riemann zeta function shows the natural way to think about the distribution of primes lies in complex analysis since the arithmetic proof that $\pi(x) = x/l(x) + o\,(x/l(x))$ as $x \to \infty$ where $l(x) = 1+1/2+1/3+\ldots+1/[x]$ does not have a natural extension in the language of arithmetic.

We hand over to the increased power of analysis to interpret the prime number distribution in terms of the zeros of the Riemann zeta function. The extension to further terms to describe the distribution of primes is seen to be naturally available in the words of analysis with the natural logarithm, the logarithmic integral and the zeros of the Riemann zeta function. Difficult as elementary proofs of the prime number may be, they are still counting exercises on the prime numbers and the theorem is not at all concerned with the specific prime structure of the natural numbers, other than the simple logical distinction that some are prime numbers and some are not prime numbers.

When we think about the convergence of $\sum \mu(n)/n^s \quad (1 \le n \le \infty)$ we have inbuilt a global property of the Möbius function with the **all** the values of the function involved in the convergence of the series in a half plane. One back from this, assuming the natural ordering of numbers, the ordering of **all** the square free numbers has a bearing on the half plane of convergence. From the point of view of analysis there may be ways of getting some information about the convergence but this is not so in arithmetic. We need to know too much about the relationships between prime products to get results which embody this knowledge. The prime factorisations and the order of the square free numbers need to be worked out before we can understand the size of the Möbius sum function in arithmetic. This activity becomes unbounded as $x \to \infty$. We do not have an inductive link in multiplicative structure to shorten this requirement. The theoretical process of estimating the

sum function by solving equations leads to a form of circularity (see appendix 1). The global connection from the equation $\sum 1/n^s = \Pi(1-1/p^s)^{-1}$ defines the limits of connection between the additive and multiplicative structure of the natural numbers. This formula is the gateway in analysis to properties of primes but it presents a wall in arithmetic which cannot be climbed. Although it is not customary in number theory to distinguish between the analytic primes and the arithmetic primes there is a distinction which should be remembered.

Section 1

Surety in explanation

The traditional explanation of axiomatic systems for the non- logician is that they produce the working material in independent assumptions so there is clarity in all the logical workings which lead to the proofs of theorems which follow from the assumptions.

A simple split in axiom statements is between working type assumptions which contain the content for the language and the thought type assumptions which are more to do with native reasoning. Peano's axiom for complete induction allows arguments to be closed off with universal quantification.

The view is taken in this discussion that universal quantification in analysis and added to incomplete arithmetic is a powerful devise in language but from a common inductive standpoint '*without exception*' has meaning but '*for all*' does not have meaning in the unbounded case. Unbounded inductive argument is accepted but no extra logical processes are involved – we do not gain the capacity for unbounded thought assuming mathematical induction– but we have the capacity to believe that a pattern with inductive structure is imaginable and the capacity to reason that acceptance of this will never lead to contradiction. We return on a number of occasions to the idea that the proofs in analysis must reduce to inductive proofs even though this may not be evident from some of the constructions. There is however, the assumption of existence in the unbounded case – thus $1+1/4 + 1/9 + 1/16 + \ldots = \pi^2/6$ assumes the existence of an entity – a series whose *convergent* appearance defines an entity π. This is certainly not an assumption of finite arithmetic. We are however, encouraged that we are able to prove in analysis that the area of a circle is πr^2. This is an analytical result because the entity π can only be understood inductively in arithmetic. It is only after the real numbers have been invented that we are able to discover that $1+1/4 + 1/9 + 1/16 + \ldots = \pi^2/6$ and this is thus not a result of elementary methods.

A problem does arise in the construction of the real numbers in that we cannot count them so the collection of them all does not have the simple inductive comprehension of the natural numbers. Through the construction process we are able to order the real numbers but this is at a theoretical level as we cannot understand consecutive real numbers. However, in mathematical activity, when we are concerned with what real numbers do, we are only concerned with processes which we are able to understand inductively through the '*without exception*' interpretation. In this light sensible activity in analysis does not hinge on anything other than inductive reasoning. For example, we can convince ourselves that $1+(1/4) + (1/9) + (1/16) + \ldots = \pi^2/6$, inductively by getting the difference between the two sides smaller and smaller but we may suspect a certain circularity in line with Poincare's objection to complete induction. But it is good circularity because this type of thing is expressing relationships between entities in the language.

We see the uses of infinity in analysis as a convenience in the language, conventions in the language which lead to a tidy ways of explaining proofs in analysis using inductive reasoning. We discuss further how it is our simple inductive reasoning in arithmetic which

leads to acceptance of analytical results. In other words the inductive assumptions of natural (non-mathematical) thought are all we ever use in doing arithmetic and analysis. Kronecker's famous quotation is about the order of things – the natural numbers help define infinity, not the other way around. But we cannot object to devices being made up if they are useful and interesting. All we need to do is ensure we stay within the realm of reasoning.

Complex analysis is a simple field extension of the real field and so we can engage in number theory in this domain accepting the sensibility of results in the developing language once we are satisfied the arguments have eliminated counter example 'without exception'.

Arithmetic, analysis and consistency

Arguments in arithmetic as defined are results in the finite realm. The development of properties of these natural numbers is simply rule following within accepted ways of reasoning. It is the availability of natural numbers which is what one needs to do this kind of number theory. From the counting numbers or the natural numbers described in a more formal way we come across the notion of unbounded and order. The ordering is in line with the pattern 1, 1+1, 1+1+1 … and is simply $a>c$ if there exists b such that $a+b = c$. Unbounded is recognition that there is no limit to the construction 1, 1+1, 1+1+1, …. .
We know from the practical experience of modern mathematics that the assumption of the infinite set has been a good idea. Nevertheless, for some logicians, it is an assumption which cannot be proved and consequently any arguments which contain this assumption may be thought like the inductive mathematical argument p(2), p(3), ….p(n) .. with p(1) missing (or at least non-verifiable). We cannot eliminate the unease of having no more than a system which is internally consistent, or at least, appears to be so, but which we cannot justify any further than this. Add this assumption into arithmetic and we have surely got to come across things we cannot prove without the assumption. The natural numbers of Peano without the need for universal quantification does not attract the unease of the more restricted theory because the development is within finite reasoning which has more to do with classical logic and cognition than concerns about formal systems. We thus proceed on the latter course relying on the consistency of arithmetic without universal qualification and accepting the constructions and theorems of complex analysis are verifiable to any degree of approximation for any specific choices of functions and variable values we choose to examine.

Arithmetic and analysis (language)

In the process of moving from simple arithmetic to complex analysis we are involved in the process of building new entities which need new words to describe them and a bigger language which becomes a muddle of the old language and all the new words and sentences from the new creations. In this process there is interest in finding out how strong the old language is relative to the new language and this is one of the past times of elementary methods. Thus we see the *elementary methods* of elementary methods in number theory as a distinction in language as much as a language with different logical structure. There is interest in understanding the results is analysis which are provable in arithmetic or which have interpretation in the arithmetic. It becomes clear that there are results in analysis which have interpretation in arithmetic which do not have proof in arithmetic.

The prime number theorem and its extension to Dirichlet's theorem are quite extraordinary in that there are both arithmetical and non-arithmetical proofs but the elementary proofs do not have the strength to get beyond 'first order' expression. If we let $l(N) = 1+(1/2)+(1/3) + \ldots+(1/N)$ we are able to prove $\pi(N) = x/l(N) + o(x/l(N))$ as $N \to \infty$ but there is no natural

arithmetical extension to higher order approximation. It is only in analysis and connection with the logarithmic integral that we find a deeper and cleaner explanation for $\pi(x)$. This then is a language problem as much as a logical problem. This apparent failure of the mother language (arithmetic) to talk about itself clearly is one of the drivers for looking at new structures beyond arithmetic. We indicate in the following notes how this crossroads in language in fact contains an explanation for the truth of the Riemann hypothesis.

As things have played out, analysis and arithmetic are competing entities, analysis forever trying to prove superiority with more and more invention but arithmetic as the mother forever holding the upper hand. The tension is perpetual because analysis is a genuine extension of arithmetic in that the language of analysis does not have complete translation from arithmetic. If the theorem in analysis has interpretation/translation in arithmetic there cannot be a contradiction to the analytical result in arithmetic. This follows directly from the interpretation that analysis is also a language extension of arithmetic where the new language is only using construction processes which are understood inductively from the base arithmetic. What we emphasise in discussing the problem of the Riemann hypothesis is the undecidability in arithmetic of certain statements related to the quasi-Riemann hypothesis.

Logical equivalence

Logical implication and logical equivalence are used in the logical working of mathematics to understand relationships. Symbolically, we have $P \rightarrow Q$ and $P \equiv Q$ for theorems P and Q.

One level back from this is the situation where P and Q have components in common but there is no simple logical relationship between them.

For example, the theorems $P(12= 2^2.3)$ and $Q(35=5.7)$ may be proved independently of each other and it requires additional components (*'something added')* to set up a framework where we prove $P \equiv Q$. In theory at least if we are given the prime numbers less than or equal to N in order we may order all the prime products, allowing for repeated primes, less than or equal to N. If we call this activity, the theorem R(N), as a mathematical activity we have P+R(N) includes Q and Q+R(N) includes P which in this context may be expressed as $P \equiv Q$. What we know determines whether we have implication or equivalence or less connection.

In the activity process R(N) we need to *'know'* the primes $p_1, p_2, \ldots p_{\pi(N)}$ to get the theorem R(N) for the value N. It should be obvious that knowing this set of primes is a minimal requirement for proving the theorem R(N).

Within human mathematical activity there is a thinker doing the thinking and we wish to think in terms of a lower bound count for the elemental thought processes which an individual needs to go through in order to believe an implication in the mathematical context is true? We do not wish to count the lines in a line by line proof but think about counting different entities and connecting thoughts which need to be located before the proof of some equivalence or implication is possible. Here, we use the counting numbers as a component in examining reasoning processes.

An idea which still needs to be defined is that of the *minimum length of a proof* in arithmetic. If we have two theorems P and Q in arithmetic with $P \rightarrow Q$, we say P is stronger than Q if the minimum length of $P \rightarrow Q$ is shorter than the minimum length of $Q \rightarrow P$. In other words something extra is required in arithmetic to get the equivalence.

Then an unbounded sequence of ever strengthening theorems in arithmetic is not provable in arithmetic because the count of a proof becomes unbounded. This is really implicit in the arguments in this note.

Axiom: (non-existence):-

If in arithmetic an unbounded number of thought processes are necessary for proof of a collection of theorems, then there does not exist a provable theorem in arithmetic which establishes the truth of these theorems.

This is taken to be an observation which cannot be false. We see it as a complement to incomplete inductive reasoning. It is not quite the negation of inductive reasoning but this assertion is on that side of things and has more to do with understanding infinity as a device as discussed in earlier sections.. We cannot get disconnected unboundedness out of human thought processes and consequently, any logical mathematical constructions in arithmetic which are able to achieve this in terms of logical consequences are not provable within arithmetic.

Definition

An unbounded sequence of theorems in arithmetic p(1), p(2), p(n) … is called non-inductive if a proof of p(1), p(2), ….p(n) necessarily becomes unbounded as $n \to \infty$.

A simple example is the R(N) theorem of the previous section, which is the derivation of the factorisations of 1,2,3,… N. There is no finite collection of inductive mechanisms which allow us to give a finite explanation of the factorisations of an ever increasing set of consecutive natural numbers. The primes need to be determined in order to compute the factorisations and the primes are unbounded.

The idea is that unbounded theorems in arithmetic are unprovable in arithmetic because of the non-existence axiom.

Unprovability and undecidability within arithmetic (finite set theory)

There are different understandings of what 'unprovable' and 'undecidable' mean.

The context for these questions in this discussion is simply about the type of explanation still possible for the Riemann hypothesis.

The main drive overall has been to look for a theoretical proof that the hypothesis is true. The view point here is that the hypothesis is posed in a language and there is the implicit hope that there is the possibility of deciding whether the hypothesis is true or false within the language. The notion of 'within the language' allows for building new structures and theories, in much the same way new words appear with new meanings in common languages. There is also the possibility of wider systems of explanation with more axioms and more explanatory power. There would not seem to be an obvious end to the process of abstraction and it has proved to be one of the very fruitful approaches to problem solving and understanding. It should be recognised that there are some limits to the things which are explainable in a language or at least limits on the nature or quality of the explanation. We use the limitations of argument in arithmetic

Let $\underline{\sigma}$ be any number in [½, 1) and let the quasi Riemann hypothesis be the statement that $\underline{\sigma}$ is the least upper bound for the real part of zeros of ζ in the critical strip.

An argument is shaped along the lines that the value of $\underline{\sigma}$ in the quasi-Riemann hypothesis can only be $\underline{\sigma} = ½$ because that result corresponds to the weakest result and any stronger choice leads to contradiction.

This argument does not exclude the possibility of a more abstract proof or abstract visualisation which is acceptable as a proof but such a proof would not appear necessary for a valid explanation. In other words, the hypothesis itself has a relatively simple explanation in language and logic and thought and does not need more abstraction to arrive at an explanation. This is almost the opposite of Fermat's last theorem where a complex abstract proof is accepted but the existence of a proof in arithmetic using roughly the amount of arithmetic available to Fermat is still undecided.

Some limitations of proof in arithmetic

Example 1

Numerical investigation into the abc conjecture exposes it as something which is possibly true but this could well be the end of matters.

Firstly, we note an analytical result.

The mapping $a \rightarrow a.\text{rad}(a)$ is clearly 1-1 and we have a 'zeta type' analytic relationship

$\sum 1/(a.\text{rad}(a))^s = \prod(1+1/p^{2s} + 1/p^{3s} + \ldots\ldots)$ but experience has shown there are limitations to the sorts of understandings which can be developed from these kinds of relationships.

We see the abc conjecture as a problem which cannot be solved positively in arithmetic because it involves unbounded verification. The process of numerical investigation may yield results which may make the possibility of its truth less likely or more likely. A disproof of the abc conjecture would be equivalent to an unbounded sequence of triples $\{a_n,b_n, c_n\}$ such that for some fixed $\epsilon > 0$ we have $\lim \text{Max}\{|a_n|,|b_n|,|c_n|\}/\text{rad}(|a_nb_nc_n|^{(1+\epsilon)}) = \infty$ as $n\rightarrow\infty$.

Now this is explicitly about the prime structure of an unbounded number of numbers but the prime structure of a number is only determined by a process of computation.

We may in fact consider the elements $\text{Max}\{|a_n|,|b_n|,|c_n|\}/\text{rad}(|a_nb_nc_n|)$ as 'essentially different' in some sense or other.

It is easier to contemplate this as being an unreasonable thing to ask of the language of arithmetic than it is to believe that a finite proof in arithmetic could release so much 'equivalent' information.

Both the positive proof and the negative rebuttal would ask for more pattern in arithmetic than the language can actually find finitely.

When problems in number theory involve a mix of additive and multiplicative concepts there is a need to sort out problems which actually are asking for an unbounded amount of verifications which are in some sense independent of each other. We only get finite proof when we find sufficient finite pattern.

Example 2

To labour these points a little further we note that we cannot 'know' the factorisation' of every natural number and we have to know quite a lot of information to determine the factorisation of a specific number. If we then define the Möbius function through the theoretical multiplicative structure of numbers and ask the question:

Is $M(x) = \sum\mu(n) = o(x^{½+\epsilon})$ as $x \rightarrow \infty$ (summation $1 \le n \le x$)? we are really asking a lot of the language of number theory in arithmetic.

It is not difficult to construct absurdly difficult questions using multiplicative structure which would not be considered worth asking. Less easy to recognise may be the questions which arrive within tidy theories, the answers to which would provide clear cut results rather than provisional results contingent on something or other being true. Such problems are favoured as being provable because of what they yield. The abc conjecture and the Riemann hypothesis are two such examples.

Interpretation of the Riemann hypothesis in arithmetic

The Riemann hypothesis is equivalent to the theorem that $M(x) = \sum\mu(n) = O(x^{½+\epsilon})$ as $x \rightarrow\infty$, (summation $1 \le n \le x$), for each $\epsilon > 0$ and with suitable adjustment to the ϵ condition we have a statement in arithmetic equivalent to the Riemann hypothesis. The equivalence is not immediately obvious but it is obvious that this result implies the Riemann hypothesis. If the arithmetical condition is met the reciprocal of the zeta function converges as a Dirichlet series for $\sigma > ½$ which rules out zeros of ζ in $\sigma > ½$.

Theorem

With summation $1 \le n \le x$,

$M(x) = \sum\mu(n) = O(x^{\underline{\sigma}+\epsilon})$ as $x \rightarrow\infty$, is undecidable in (½, 1) in arithmetic.

Proof

In arithmetic we need to **recognise** that a proof of an estimate for $M(x)$ must involve knowing more than an amount equivalent to the order and the numerical value of the primes less than or equal to x. No matter how large we choose x to be, we will always be faced with a finite collection which we need to order in terms of both the multiplicative structure of a number and its magnitude relative to the other numbers. As we increase the size of x, new structures (products with k distinct primes) come into play and the ordering of these new structures needs to be related to the ordering of the earlier structures. This added dimension lifts the problem of an estimate from a counting problem to a non-inductive problem. The recognition required here is recognition of non-pattern rather than the familiar pattern recognition required in mathematical induction.

If the restriction to arithmetic is lifted to analysis we are in a realm where the 'global' Möbius function appears in the language via such things as $1/\zeta(s) = \sum\mu(n)/n^s$ and a proof of the Riemann hypothesis may be possible in this realm because of the increased strength of mathematical analysis.

Corollary

All computed zeros of the Riemann zeta function in the critical strip will be simple and lie on $\sigma = ½$.

For practical purposes in computation we are just concerned with things matching up to any specified degree of accuracy. We then take the notion of ϵ -convergence through the definition (for example) that f(x) is ϵ convergent to l as $x \rightarrow a$ if there exists δ $(=\delta(\epsilon))$ such that $|f(x)-l| < \epsilon$ whenever $|x-a| < \delta$.

We could talk in terms of ϵ convergence (where we have convergence in the analytical sense) rather than convergence but it would seem a tedious task to develop classical analysis in this language. It is however an assumption in computational methods that analysis is the natural extension of arithmetic and the results of computational methods are to be believed. In the calculation of the zeros of the Riemann zeta function, the rules for counting the zeros and checking on the multiplicity of zeros are written down and may be followed through 0-1 type logic. The outcomes may be reported without needing to know the interpretation in the theory of the Riemann zeta function. This activity is essentially an activity in arithmetic as there is no need to reference an axiomatic system. The computer, the programmers and the output extend the type of arithmetic possible.

We then examine why the undecidability in arithmetic of $M(x) = O(x^{\underline{\sigma}+\epsilon})$ as $x \to\infty$ with $\underline{\sigma} \in (½, 1)$ leads to $\underline{\sigma} = ½$.

If a zero were located off the line at say a+ib with a> ½ then the above statement would have been decided in arithmetic in (½, a).

This contradiction in arithmetic proves that all the zeros in computation lie on $\sigma = ½$.

Further, since $M(x) = O(\sqrt{x})$ as $x \to\infty$ is a stronger statement than $M(x) = O(x^{½+\epsilon})$ as $x \to\infty$ this statement is also undecidable in arithmetic. Since a calculated multiple zero would negate the modified Merten's conjecture, it follows that the calculated zeros will be simple zeros, Odlyzko and Riele [5].

Notes:

Someone in the back row says – it's all well that it is obvious but to get a proof rather than an explanation you really need to 'prove' the limits of inductive explanation in multiplicative structure.

The problem here is that there is not much from the past we have seen that helps as the orientation seems to be from the other end of things. There is a prime between N and 2N and even this requires effort. We can describe the prime structure of N! but these numbers are rare in the collection of natural numbers. The fact that the Möbius function only takes three values does not simplify the problem because an arithmetical interpretation of the hypothesis is that it says something about the multiplicative structure in terms of ordering.

We note that the estimate $M(x) = o(x)$ as $x\to\infty$ is equivalent to the prime number theorem in the form $\pi(x) = x/\ln(x) + o(x/\ln(x))$ as $x \to\infty$. We have noted that apart from a trivial way this is not a theorem about the prime structures of the natural numbers and how they inter-relate. This difference correlates with the line $\sigma = 1$ in the theory of the Riemann zeta function and this is why it is impossible to 'move inside' $\sigma = 1$ in arithmetic. We move from a question about the simple count of primes to a question about the ordering of the multiplicative structures of the primes and these are quite different sorts of questions.

The argument to date is simply that there is no inductive link to enable a decision on the value of $\underline{\sigma} \in (1/2, 1)$ for $M(x) = O(x^{\underline{\sigma}+\epsilon})$ as $x \to\infty$ in arithmetic.

If we try to solve M(N) from the system of equations (see appendix a)

$\sum\mu(k)[n/k] = 1$ $(1 \le k \le n,\ 1 \le n \le N)$ we just end up in circularity with the last value equal to M(N) (see appendix 1) and for any N we only have an $O(\sqrt{N})$ numbers in the coefficients which places a serious restriction on the coefficients.

The idea in arithmetic is that a proof must be unbounded and hence is impossible.

Another way of looking at things is through the sieve of Eratosthenes where with some colouring and some counting we could look at the square free numbers and calculate M(x) or if we wished the corresponding sum function for the Liouville function S(x). If there is no inductive pattern in the primes how could there be inductive pattern in this much more complicated structural question. There is no reason to suppose assigning -1 to numbers with an odd number of primes and +1 to the other numbers that somehow inductive pattern is restored.

The sort of inductive link in ordered prime structure which would provide a argument in arithmetic for the Riemann hypothesis may possibly imply that the infinite series $\sum\mu(n)x^n$ could be expressed as a simple closed formula recognisable in arithmetic but in practice this occurs only with series using the principle of 'pattern in, pattern out' such as $\sum nx^n$. The reader may be interested in Solomon Golomb's [4] series related to the twin prime problem

It is interesting what comes out of an asymptotic estimate for $p_{n+1} - p_n$ for consecutive primes. We may wonder in arithmetic if we assumed $p_{n+1} - p_n = O(n^\epsilon)$ as $n \to \infty$ if it would make an elementary proof of the Riemann hypothesis look any easier. Even this assumption is not revealing much about the order of multiplicative structure.

At a point where the individual is convinced beyond any doubt of the outcome regardless of the imperfect nature of the explanation a question arises about the usefulness of computing further zeros. It is akin to a numerical search to find non-uniqueness of factorisation in the natural numbers and we know that a computer search will never find a counter example. It may even be that the two things are equivalent in that non-uniqueness may lead to exceptional zeros. If it is the case in physical science that close estimates for the imaginary parts of zeros are useful in some modelling process there may be sense in further computation. In terms of extending computer scope there are plenty of other problems to be interested in.

Section 2 – Explanation 2

The axiom we need in arithmetic is along the lines that if P, Q are theorems in arithmetic and $P\to Q$ on a class of functions but $Q\to P$ requires addition information, then the shortest proof of $P\to Q$ is less than the shortest proof of $Q \to P$. In previous notes P and Q were called essentially different theorems. The finite proof finite theorem assertion (FPFTA) is that if an unbounded number of theorems are essentially different in a logical hierarchy in arithmetic there is no provable theorem in arithmetic from which they all follow.

The theory of the Riemann zeta function may be used to set up such a hierarchy in arithmetic. Some results are developed which allow another view of the Riemann hypothesis. The techniques have been in the literature in fragmented form for many years.

Notation and usage and conventions:

Generally, if $A(x) = \sum a(n)$ $(1\le n\le x)$ then we also write $A_1(x) = A(x)$ and $A_k(x) = \sum A_{(k-1)}(n)$ $(1\le n\le x)$. We call the $A_k(x)$ the higher summation functions. Although we have a starting function $A_1(x)$ this is somewhat relative as we could start the sequence from any of the sums. We could also build $A_0(x)$, $A_{-1}(x)$, $A_{-2}(x)$ … …

All integrals are 1 to ∞.

All Dirichlet series sums are 1 to ∞.

All summations of simple number theoretic functions are over natural numbers $\le x$.

Two theorems which are assumed in the following discussion:

1. Let $f(s) = \sum a(n)/n^s$ where $A(x) = \sum a(n)$. Then $f(s) = \int A(x)/x^{(s+1)}\, dx$ (the integral representation for Dirichlet series), Titchmarsh [7].
2. If the a(n) are real and eventually of one sign then the function represented by the series has a singularity at the real point on the line of convergence of the series, Titchmarsh [6].

We extend the Landau notation o, O, $\Omega_{+,-}$ to |o, |O, $|\Omega_{+,-}$ where we wish to indicate the precise nature (either assumed or already proven) about the behaviour of a function within the limitations of these notions.

Let $M_1(x) = \sum \mu(n)$ and for K > 1, $M_K(x) = \sum M_{(K-1)}(n)$ where μ is the Möbius function

Theorem 1

For K ≥ 1 let $L_K(s) = \sum M_K(n)/n^s$.

$L_K(s) = 1/(s-1)(s-2)\dots(s-K)\zeta(s-K) + E_K(s)$ where $E_K(s)$ is analytic for σ > K.

Proof

We note the trivial estimate $M_K(x) = O(x^K)$ as $x \to\infty$.

The proof is by induction. The method may be used to verify the theorem for K= 1.

$L_{(K+1)}(s) = \sum M_{(K+1)}(n)/n^s = s \int \{\sum M_{(K+1)}(n)\}/x^{s+1}\, dx$

$= s \int\{\sum[x]-n+1)M_K(n)\}/x^{s+1}\, dx$

$= s \int \sum M_K(n)/ x^s\, dx - s \int \sum nM_K(n)/ x^{s+1} + p_K(s)$ where $p_K(s)$ is analytic for σ > K+1.

Thus $L_{(K+1)}(s) = \{(s/(s-1))L_K(s-1)\} - L_K(s-1) + p_K(s)$

$= \{1/(s-1)\} L_K(s-1) + p_K(s)$ and the result follows.

We use this result to derive the main results about the oscillatory behaviour of the Möbius sum function and the higher summation functions.

As in section 1, let $\underline{\sigma}$ be the smallest real number such that ζ(s) ≠ 0 for $\sigma > \underline{\sigma}$.

Theorem 2

$\mathbf{M}_K(x) = |\Omega_{+-}(x^{K-1+\underline{\sigma}-\epsilon})$ as $x \to\infty$.

Proof

Suppose $M_K(n) + An^{(K-1+\underline{\sigma}-\epsilon)}$ is eventually of one sign, where A is a non-zero integer.

Then the function defined by the Dirichlet series, $H_K(s) = \sum(M_K(n)+ An^{(K-1+\underline{\sigma}-\epsilon)})/n^s$ has a singularity at the real point on its line of convergence.

It follows from the preceding theorem that

$H_K(s) = 1/[(s-1)(s-2)\dots(s-K)\zeta(s-K)] + A\zeta(s-K+1-\underline{\sigma}+\epsilon) + E_K(s)$ where $E_K(s)$ is analytic for σ > K.

Moving from right to left along the real axis we find the first singularity of $H_K(s)$ at $s-K+1-\underline{\sigma}+\epsilon = 1$.
i.e $\sigma = K+\underline{\sigma}-\epsilon$.

Since $H_K(s)$ is then analytic for $\sigma > K+\underline{\sigma}-\epsilon$ it follows that $\zeta(s-K)$ is analytic for $\sigma > K+\underline{\sigma}-\epsilon$. In other words $\zeta(s)$ is analytic for $\sigma > \underline{\sigma}-\epsilon$. This contradicts the choice of $\underline{\sigma}$.

The fact that this is the best possible result in both the positive and negative direction separately also follows from the statement theorem.

We also have the unconditional analytical results that $M_K(x) = \Omega_{+-}(x^{K-1/2-\epsilon})$ as $x \to\infty$ for K=1,2 3…, since $\underline{\sigma} \geq \frac{1}{2}$. It should also be clear that each of the statements of the theorem (K=1 or K=2 or ....) is equivalent to the quasi-Riemann hypothesis.

Let
$P_K(x, \underline{\sigma}) \equiv [\, \mathbf{M}_K(x) = |\,\Omega_{+-}(x^{K-1+\underline{\sigma}-\epsilon})$ as $x \to\infty]$ (K = 1,2,3.....) ....(1)
and unconditionally
$Q_K(x) \equiv [\, \mathbf{M}_K(x) = |\,\Omega_{+-}(x^{K-\frac{1}{2}-\epsilon})$ as $x \to\infty]$ (K = 1,2,3.....).............................(2).

We now use the finite proof, finite theorem assertion to establish the unprovability of (2) and (1) in arithmetic.

Firstly, we do require there is no inductive links in the language of arithmetic between the statements in (1) and that there is no inductive links in the language of arithmetic in the statements embodied in (2). **Without** the analytical theory of the Riemann zeta function we do not have a general theorem that $Q_K(x)$ follows from $Q_1(x)$, $Q_2(x)$, ...$Q_{K-1}(x)$.

It is a peculiar property following from the nature of the Möbius function in the analytical theory and the strength of the language of analysis which allows these links between analytic results and statements which have interpretation in arithmetic.

In language of elementary arithmetic, we cannot establish the truth of these statements. A similar comment applies to the unbounded statements $P_1(x, \underline{\sigma})$, $P_2(x, \underline{\sigma})$, ...$P_K(x, \underline{\sigma})$ .... and ignoring the special meaning of $\underline{\sigma}$, establishing $P_{K+1}(x, \underline{\sigma})$ is hierachically more difficult that establishing $P_K(x, \underline{\sigma})$, since $P_{K+1}(x, \underline{\sigma})$ implies $P_K(x, \underline{\sigma})$ but the statement $P_K(x, \underline{\sigma})$ implies $P_{K+1}(x, \underline{\sigma})$ requires more input about the Möbius function. The statements $P_K(x, \underline{\sigma})$ are thus unprovables in arithmetic (K = 1,2,3 ...) Now the weakest unprovable in arithmetic is thus $P_K(x, \frac{1}{2})$ for each K.

From this we conclude that the Riemann hypothesis is unprovable in simple arithmetic assuming the FPFT axiom. Since the computation of zeros on $\sigma = \frac{1}{2}$ is an activity in arithmetic we will not be able to find a zero off the line $\sigma = \frac{1}{2}$. Indeed, this would establish the truth of an unprovable in arithmetic stronger than the Riemann hypothesis and would thus contradict the truth of the Riemann hypothesis.

We have discussed in the earlier section that the unprovability of the modified Merten's conjecture in arithmetic implies the simplicity of the zeros.

Appendix 1: Estimating the Möbius sum function in arithmetic

For convenience we define the Möbius function μ by $\sum\mu(n)[N/n] = 1$ for $N \ge 1$, where summation is $1 \le n \le N$ and $[N/n]$ is the largest integer less than or equal to N/n. It is curious that this definition puts one version of the Riemann hypothesis only a few minutes away in terms of a problem description.

$M(x) = o(x)$ as $x \to \infty$ is provable by elementary methods and is equivalent to the prime number theorem, and gets a first order description for a count of the prime numbers in arithmetic.
We examine a pointer towards thinking that stronger estimates for $M(x)$ may not be possible in arithmetic.
If we think in terms of linear equations, the above definitions for the μ function provide a way of calculating the sum function $M(x) = \sum\mu(n)$ without calculating the individual $\mu(n)$.

Indeed, with N = 3, 2 and 1 we have

$3\mu(1) +1\,\mu(2) +1\mu(3) = 1 = P(3)$
$2\mu(1) + 1\mu(2\,) \qquad = 1 = P(2)$
$1\mu(1) \qquad\qquad\quad = 1 = P(1).$

From the LHS we have $P(3) - 2P(1) = \mu(1)+ \mu(2)+ \mu(3) = M(3)$…………………(1)

Hence, $1 - 2.1 \quad = M(3)$.

i.e. $M(3) = -1$.

Further numerical investigation will reveal that this procedure may be continued indefinitely and we use the system of equations

$1 = P(N) = \sum\mu(n)[N/n]$ for $N \ge 1$, to reduce the coefficients of $\mu(N)$, $\mu(N-1)$, $\mu(N-2)$.. to unity.

We end up with an equation of the form

$P(N) - a(2)P(N-1) - a(3)P(N-2) - \ldots\ldots - a(N)P(1) = M(N)$ where $a(2)$, $a(3)$ … are integers.

The number of distinct $P(r)$ for $1 \le r \le N$ which appear in these equations is in fact about $2[\sqrt{N}]$, corresponding to $[N/1]$, $[N/2]$, …..$[N/[\sqrt{N}\,]$ and 1,2,3 ……$[\sqrt{N}]$ or thereabouts. See Gelfond and Linnik [3].

To see the pattern of the general reduction it is convenient to use a vector notation.

$\underline{P}^*(N) = (\mu(1)[N/1], \mu(2)[N/2], \mu(3)[N/3], \ldots\ldots\ \mu(N)[N/N], 0, 0, \ldots..)$ and
$\underline{M}(N) = (\mu(1), \mu(2), \mu(3), \ldots\ldots\ \mu(N), 0, 0, \ldots..)$, where as usual μ denotes the Möbius function and $M(N)$ denotes the sum $\sum\mu(n)$ with summation $1 \le n \le N$.

We use the usual vector addition and scalar multiplication here.

Theorem 1

$\underline{M}(N) = \sum\{M(N/n) - M(N/(n+1)\}\underline{P}^*(n)$ with summation $1 \le n \le N$.

This is the unique reduction process described above.

Proof

We look at the scalar factor associated with $\mu(k)$ in each side of the equation.

On the LHS the scalar factor is 1.

On the RHS the scalar factor is

$\sum[r/k]\{M(N/r)-M(N/(r+1))$ (summation $k \le r \le N$)

$= \sum[r/k]M(N/r) - \sum[r/k]M(N/(r+1))$ (summation $k \le r \le N$)

$= \sum[r/k]M(N/r) - \sum[(r-1)/k]M(N/r)$ (summation $k \le r \le N$)

$= \sum\{[r/k] - [(r-1)/k]\}M(N/r)$ (summation $k \le r \le N$)

$= \sum M(N/tk)$ (summation $1 \le t$)

$= 1.$

So the numerical approach outlined in the introduction has M(N) as the final adjustment needed in the step wise reduction of
$\underline{P}^*(N) = (\mu(1)[N/1], \mu(2)[N/2], \mu(3)[N/3], \ldots\ldots\ \mu(N)[N/N], 0, 0, \ldots..)$ to

$\underline{M}(N) = (\mu(1), \mu(2), \mu(3), \ldots\ldots\ \mu(N), 0, 0, \ldots..)$ moving from right to left.

Whereas this provides a systematic way of calculating M(N) there is not a direct theoretical application as we just end up with M(N) is an estimate for M(N).

In other words the solving of the system of equations by this method – in theory – just arrives back at the quantity rather than a new expression from which it may be possible to get a desired estimate.

This circularity gives a clue about why it may be impossible to decide the order of M(x) in the form $M(x) = O(x^{(a+\epsilon)})$ as $x \to \infty$ for $½ \le a \le 1$.

In the process of reducing
$\underline{P}^*(N) = (\mu(1)[N/1], \mu(2)[N/2], \mu(3)[N/3], \ldots\ldots\ \mu(N)[N/N], 0, 0, \ldots..)$ to

$\underline{M}(N) = (\mu(1), \mu(2), \mu(3), \ldots\ldots\ \mu(N), 0, 0, \ldots..)$ we are essentially dealing with vectors of length N but we are restricted in arithmetic to use only $2[\sqrt{N}]$ specific vectors. The distinct numerical values of $[N/n]$ $1 \le n \le N)$ are the only ones which come into the reduction process. The restriction to about $2[\sqrt{N}]$ of N possible numbers in the equations which we have available is a structural restriction in any numerical manipulation in getting an estimate for M(x). There don't seem to be enough numbers to work with.

The relationship between the Möbius function and the greatest integer function is of such a nature that it does not seem possible to separate them sufficiently to get an improved estimate for the Möbius sum function.

If we sum the equations $\mu(m) \sum\mu(n)[N/nm] = \mu(m)$ $(1\le n \le N, 1\le m \le N)$
with some rearrangement we see
$M(N) - 2M(\sqrt{N}) = -\sum\sum \mu(n)\, \mu(m)[N/nm]$ summation $(1\le n \le \sqrt{N}, 1\le m \le \sqrt{N})$.

This may be written

$M(N) - 2M(\sqrt{N}) = -Ng(\sqrt{N})^2 + \sum\sum \mu(n)\, \mu(m)\{N/nm\}$ summation $(1\le n \le \sqrt{N}, 1\le m \le \sqrt{N})$,
where $g(N) = \sum\mu(n)/n$ $(1\le n \le N)$ and $\{N/k\} = N/k - [N/k]$.
This relationship is quite useful for calculating M(N) over large ranges in computer investigation because M(N) is calculated only using the μ values up to $[\sqrt{N}]$.

We also see here that M(N) may be expressed as a complicated mix of the first $[\sqrt{N}]$ μ values and the distinct values of $[N/k]$ $(1\le k \le N)$ which number about $2[\sqrt{N}]$. Also, because of the known oscillatory properties of M(N), there are unbounded times where
$\sum\sum \mu(n)\, \mu(m)\{N/nm\}$ summation $(1\le n \le \sqrt{N}, 1\le m \le \sqrt{N})$ is the dominant (positive) term in the equation $M(N) - 2M(\sqrt{N}) = -Ng(\sqrt{N})^2 + \sum\sum \mu(n)\, \mu(m)\{N/nm\}$
summation $(1\le n \le \sqrt{N}, 1\le m \le \sqrt{N})$.

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