A note on the Riemann Hypothesis (RH) – Peter Braun

Abstract

This paper discusses the possibility of extending inductive reasoning to include accepting RH as a true statement in mathematics and more generally. Immediate consequences are mentioned.

Background

The author came across RH in 1963 by assigning the value -1 to each prime number and discovering the Liouville function. The nature of the sum function was contemplated at the time but it was four or five years later through reading that the connection with the Riemann hypothesis was understood.

In recent times the author has come to the view that it is remarkable that the classical Mobius sum function of Merten's conjecture gets as 'large' as it does in both positive and negative values.

Imagine the problem:

$M(x) > x^{\Delta}$ unboundedly as $x \to\infty$ and $M(y) < -y^{\Delta}$ unboundedly as $y \to\infty$ for some Δ satisfying $0 < \Delta < ½$ without the theory of the Riemann zeta function. It may be that finding Δ above 0 without the theory (approaching from below) is just as difficult as finding Δ smaller than 1 (approaching from above). A large research push seems to have been directed by the trivial upper bound estimate $M(x) = O(x^1)$ as $x \to\infty$. As we know, $M(x) = o(x)$ as $x \to\infty$ is logically equivalent to the famous prime theorem.

Indeed, this 'direction' is very much part of the number theory realm in line with refining estimates.

Introduction

The Riemann hypothesis is taken to mean that the non-trivial zeros of the classical zeta function all lie on $\sigma = ½$.

The reader is referred to Borwein, Choi, Rooney and Weirathmueller [3] for a comprehensive account of the developments around RH since inception. There is no obvious place to start or stop references so the attempt is made to limit references to a few, each of which contains many references. Suprisingly, Borwein, Choi, Rooney and Weirathmueller [3] does not give great emphasis to the oscillatory behaviour of important number theoretic sum functions perhaps as they are perceived as likely to be 'after the event 'properties. Wider references in this area may be found in Odlyzko and Riele [8], and Braun [4]

RH is accepted in number theory as the sort of problem which may require a shift in thinking in order to understand the underlying obstacles. We attempt to expand the principle of mathematical induction (local induction) to a wider realm (global induction).

By local inductive methods we mean the axioms of Peano and all the bits and pieces which are currently accepted as being necessary for the generation of true results in number theory.

Preliminaries

The order of proceeding is intended to add persuasion to the discussion and with this in mind the order of proceeding is to outline:-:

- (α) acceptable mathematical results used in the discussion
- (β) some consequences of the intended result that RH is true
- (γ) a sketch of the possible requirements necessary to obtain the intended result

The mathematics section requires experience in the applications of the theory around functions of a complex variable (since RH is expressed here in this domain), and a good level of understanding of the higher arithmetic.

(α) The mathematical results

Unless explicitly mentioned, Σ will mean summation $1<=n<=X$.

(A) Let $\{ P_1 \equiv [M_1(X) = O(X^{(1/2)+\varepsilon})$ as $X\to\infty]\}$ be the classical statement about the Möbius sum function, and let $\{P_K \equiv [M_K(X) = O(X^{(K-(1/2))+\varepsilon})$ as $X\to\infty$, $K>=2]\}$, be the corresponding statement for higher sums, where $M_K(X) = \Sigma M_{(K-1)}(n)$ ....(1).

We know, without too much difficulty that each statement is equivalent to the Riemann hypothesis.

Indeed, more generally, with

$$A(X) = \Sigma a(n)$$

we see that

$$\Sigma A(n) = [X]A(X) - \Sigma na(n) + \text{Error}(X) \qquad ....(2)$$

Perron's integral formula for Dirichlet series allows an inductive proof that the statements each imply RH.

The implication in the other direction involves contour integration using the Mellin transformation and the estimate $1/\zeta(s) = O(t^{\varepsilon})$ as $t\to\infty$. (See for example Titchmarsh [9]). The method is applicable for any natural number K.

(B) By the modified Merten conjecture we mean the statement

$M_1(X) = O(X^{1/2})$ as $X\to\infty$.

We note here that there are unbounded instances of $|M_1(X)|>A\sqrt{X}$ as $X\to\infty$ for some $A>1$, Odlyzko and Riele [8].

(C) We can show that for any $K >=1$ the statement

$M_K(X) > AX^{\alpha+K-1}$ and $M_K(Y) < -AY^{\alpha+K-1}$ for arbitrarily large A and unbounded X and unbounded Y as $X\to\infty$ and $Y\to\infty$

is equivalent to -

$\zeta(s) = 0$ in the half plane $\sigma > \alpha$ for some value of s. We denote this assertion by RH(α).

These statements may be proved in much the same way as the statements

$M_K(X) = O(X^{(K-(1/2))+\varepsilon})$ as $X\to\infty$.

Note earlier comments for references in this area.

The larger the α we choose to take the more 'amplitude' there is in the oscillatory behaviour of the sum function $M_K(X)$ as $X\to\infty$. A result which is being assumed is $1/\zeta(s) = o(t^{\varepsilon})$ and $\zeta(s) = o(t^{\varepsilon})$ as $t\to\infty$, for $\sigma > \alpha$. The method is in Titchmarsh [9].

There should not be anything in this section which is contentious. Although the results do require a certain level of understanding in the mathematics around RH they do not require more than conventional mathematical manipulation.

(β) Consequences which would flow from the proof

The suggested frame of mind here is to assume for a while that RH is unprovable.

We note first that if a proposition is unprovable then assuming it either true or false, a certain form of mathematical reasoning will at most lead to some other proposition which is also unprovable.

We assume the Riemann hypothesis is unprovable.

The modified Merten's conjecture implies the simplicity of the zeros of $\zeta(s)$, Odlyzko and Riele [8]

RH is equivalent to the statement $M_1(X) = O(^{X(1/2)+\varepsilon})$ as $X\to\infty$.

With RH established as unprovable then [$M(X) = O(\sqrt{X})$ as $X\to\infty$] is a stronger statement than RH and is also unprovable. As this conjecture is explicit in form it will forever be neither true nor false.

If a counter example to the simplicity of the zeros of $\zeta(s)$ were found it would establish the falsity of the modified Merten's conjecture. Since this is not possible, it is not possible to refute the statement that the zeros of $\zeta(s)$ are simple.

Thus any construction which produced an exceptional zero within finite arithmetic would contain a logical error.

This flow which follows from the assumption concerning RH and some known mathematical results is very appealing as it resolves difficulties around the quest for non-simple zeros of $\zeta(s)$ and the likelihood or otherwise of the modified Merten's conjecture. It also means that finding a repeated zero would imply the existence of a finite proof to decide the Riemann hypothesis.

A distraction

To continue this line a little further, some problems in arithmetic which currently remain unsolved such as the twin prime problem and Goldbach's conjecture, where conventional mathematical work appears to be converging on a solution, but never quite reaching the precise answer, may be amenable to the same approach. What would be required are logical statements with a similar sort of structure to those described in the following discussion.

Hardy's original classical approach, extended by Levinson, Conrey and others, may possibly be capable of being sharpened indefinitely and forever converging on a complete proof but never quite getting to the last full stop.

Let $T_1(X)$ = [1,0,-1,0,1,2,1,0,-1,-2, -1, 0,1,2,3,2,1,0,-1,...] be the function values on N, ($T_1(15) = 3$))

To say $T_1(X)$ is the ideal curve to think about when contemplating $M_1(X)$ is tempting as $T_1(X) = O(\sqrt{X})$ as $X \to\infty$. $T_1(X)$ is always heading towards a new maximum or a new minimum with no dawdling along the way. Thus if M(X) is balanced in the long run in achieving new maxima and new minima we may think about $M_1(X) = O(\sqrt{X})$ as $X \to\infty$ as feasible. Grotesque counter examples are unfortunately, easily available. However, the thought of developing more language around this sort of oscillating function is quite appealing for purely descriptive purposes.

Finally, a simple picture which fits with RH unprovable is to see the lines $\sigma = ½$ and σ $=1$ meeting at infinity, thus making it impossible to distinguish between the two cases in analysis. This makes RH(α) just as difficult as RH and since it is the negation of RH it is what we would expect. If the critical strip could be stretched out in some transformation, it may be easier to distinguish between $\sigma =1/2$ and $\sigma =1$.

Rex Croft at the University of Waikato, NZ calculated $M_1(X)$ up to 10^{11} in 1981-82 using the summation formula

$$\sum\sum\mu(n)\mu(m)\,[N/(nm)] \quad = \quad -M(N) + 2M(\sqrt{N})$$

where each summation is up to $\sqrt{N}$ for each variable (unpublished graphs).

At the time this gave a very exciting glimpse at M(N)'s long journey.

(γ) Concluding comments

Is a consistent extension of inductive reasoning possible?

In this section we sketch some details to support belief that an extension is achievable.

It may look as if the arguments are shaped to provide a solution to RH and this is indeed true because the nature of the original inquiry is to seek out new necessary conditions for RH using an extension to inductive reasoning. However, if the extension is consistent with the base construction and has wider application it may become difficult to refute.

An extension to a problem in logical thought involving mathematical induction, which could not be refuted in a bounded number of statements, would carry with it a certain protection. It would in a sense have 'god' like properties and with the acceptance of a new axiom it may be accepted as a legitimate method of arriving at a truth proof or a false proof – the choice depending on context.

This would not seem to involve more than conventional mathematical induction plus an assurance that there was nothing included which was false plus of course, an axiom to legitimise the process. In this way, a problem which was undecidable could be taken to be true or false depending on other contextual matters.

In a similar way, a problem which required the legitimacy of an unbounded number of logically 'different' propositions for it to be true could be taken as exactly one of true or false using a consistent extension of inductive reasoning, provided the decision on which it was going to be (either true or false) was able to break the nexus of the symmetry found in the base logic.

A clue on why it may be possible to break such a nexus as described above may be seen in a simple example.

Consider the problem of proving

$$\sum n \neq (1/2)N(N+1) \text{ for some counting number N.}$$

If we were not allowed to go through mathematical induction we would, after looking at the evidence, try to argue that an extension to reasoning was required.

In such an extension we may ask for something like Peano's axioms.

We would argue that we would have to go through the truth of an unbounded number of statements to secure the truth.

But if we allow inductive reasoning, the statement becomes a theorem, albeit a tautology.

In this sense, a construction which has the look of something 'made up' may be necessary to solve RH.

An extension of mathematical induction?

Can the axioms be extended to widen the class of propositions which may be taken to be exactly one of true or false?

What decisions need to be made to arrive at a sound extension?

The discussion puts forward suggestions which fit an explanation of RH.

We seek to find a line of reasoning which fits the problems and then turn back to examine the argument.

The simple rule is that if RH is asserted as true or not false then we must be convinced beyond any doubt that no exception to the assertion will ever be found.

A first axiom (choice) allows the existence of an inductive number τ which is not a counting number.

The first construction is for the counting numbers N (as usually understood).

The second construction then is $N_U\{\tau\}$, with the acceptance of an extension of Peano's axioms to an inductive number τ, which is not a counting number and such that $N_U\{\tau\}$ is an irrefutable extension of the natural numbers.

This may be formulated in a number of ways.

The interpretation required in the extension is that a true inductive statement in the local arithmetic (N without extension) (which really only admits truth for counting numbers) remains true in the extension (the global arithmetic). – True inductive propositions are now truly true for all natural numbers. The first assertion is that this is a consistent extension of the counting numbers. The existence of τ may be available using the axiom of choice as indicated.

Alternatively, if this causes problems, reject the axiom of choice in the base construction and replace the extended construction of N with the one outlined. Then and only then admit the axiom of choice. This insists on order being important but seems unnecessary.

Introducing τ to theorems

We refer the reader to the section labelled (1).

Please view a sequence of equivalent propositions one of which contains the number τ.

In the extension then RH is unprovable because

$P_\tau \equiv [M_\tau(X) = O\ (X^{(\tau-(1/2))+\varepsilon})$ as $X\to\infty$] is unprovable.

This leaves a situation where the ramifications of choosing RH true or RH false may be contemplated.

The next stage is to establish that P_τ may be chosen to be true. All we need here is something which will make assuming P_τ true undeniable and the appropriate choice.

From (1) we have

$P_1 \equiv P_2 \equiv P_3 \ \dots \ \equiv P_\tau$ There is a very clear way of supporting the assertion that

$P_1 > P_2 > P_3 \dots \quad \dots > P_\tau$.

That is to say, we easily find examples where if we substitute the general $A_K(X)$ in place of $M_K(X)$, we may still say

'P_1' $\to$ 'P_2' $\to$'P_3'$\to$.............. $\to$'P_τ '

but there exist examples where 'P_2' $\to$'P_1' is false, 'P_3'$\to$'P_2' is false etc.

Thus P_τ is at the end of an ever weakening chain of equivalent propositions P_1 , P_2, P_3 ... P_τ .

The conclusion we draw is that P_τ true is the appropriate choice in this context.

The proposition chain is ever weakening and because it is unprovable numerical computation will never produce a counter example.

If the Riemann hypothesis were chosen to be false we would be left with the problem of not being able to produce a counter example since a counter example through computation would prove P_τ false.

It follows from the equivalences that RH true is the appropriate choice.

We have an example of a proposition in arithmetic which is unprovable but which may be taken to be true without contradiction. Computational evidence will always support this choice.

Coming at the problem from the statements about the falsity of RH we find a 'divergent' set of propositions described in (C) if RH is assumed to be false. The value of this divergent series will be false using a similar convention. We thus have consistency if we take RH to be true. This is indeed wishful thinking but based on irrefutable evidence because that is how things have turned out. We can be as confident that RH is true as we are that $\sum n = (1/2)N(N+1)$ is true.

It also prevents 'havoc' in the distribution of prime numbers as envisaged by Bombieri [1].

Suspicion of this type of explanation may come from doubting that the generalised hypotheses in algebraic number theory may not be understood by this method. However in the next draft of this discussion we see the possibility that if RH is established as a problem which does not admit finite proof then this realisation may provide sufficiency for these generalisations to also be undecidable. It would require showing the structural similarities were driving the undecidability.

In considering the undecidability of this problem, there may be some interest in the following definition of the Mobius function using the greatest integer function.

Let $\mu(1) = 1$.

For $N \geq 2$:

If $\Sigma\, \mu(n)[N/n] = 0$ then $\mu(N) = +1$,

If $\Sigma\, \mu(n)[N/n] = 1$ then $\mu(N) = 0$,

If $\Sigma\, \mu(n)[N/n] > 1$ then $\mu(N) = -1$,

where summation is to N-1.

In arithmetic then the problem of the order of the sum function starts to look difficult.

The prime number theorem is equivalent to proving $M(X) = o(X)$ as $X \to \infty$, and proving this by elementary methods provides some indication of how difficult an explicit proof of the Riemann hypothesis would be in arithmetic.

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