Oscillatory behaviour of the Möbius function by P. Braun.

These results are used in later discussions.

Conventions:

All integrals are 1 to ∞.

All Dirichlet series sums are 1 to ∞.

All summations free from variables are over natural numbers $\leq x$.

Wherever ϵ appears in text it is assumed to be any arbitrary real positive number greater than zero.

If F is a real valued function defined for positive real numbers then $F(X) = \Omega_{+-}(X^a)$ as $X \to \infty$ means the existence of positive numbers a, b and increasing sequences of positive natural numbers $\{x_i\}$, $\{y_i\}$ where $\lim x_i = \lim y_i = \infty$ as $i \to \infty$ for which $F(x_i) > ax_i^a$ and $F(y_i) < -by_i^a$ for i = 1,2,3 ….

Oscillatory results for summation functions related to the Möbius and Liouville functions were given in Braun [1]. At the time there were fragmented results in the literature.

Two theorems which are assumed:

1. Let $f(s) = \sum a(n)/n^s$ where $A(x) = \sum a(n)$. Then $f(s) = \int A(x)/x^{(s+1)}\,dx$ (the integral representation for Dirichlet series).
2. If the a(n) are real and eventually of one sign then the function represented by the series has a singularity at the real point on the line of convergence of the series.

Let $M_1(x) = \sum\mu(n)$ and for $K > 1$, $M_K(x) = \sum M_{(K-1)}(x)$ where μ is the Möbius function

Theorem

For $K \geq 1$ let $L_K(s) = \sum M_K(n)/n^s$.

$L_K(s) = 1/(s-1)(s-2)\ldots(s-K)\zeta(s-K) + E_K(s)$ where $E_K(s)$ is analytic for $\sigma > K$.

Proof

We note the trivial estimate $M_K(x) = O(x^K)$ as $x \to \infty$.

The proof is by induction. The method may be used to verify the theorem for K= 1.

$L_{(K+1)}(s) = \sum M_{(K+1)}(n)/n^s = s\int \{\sum M_{(K+1)}(n)\}/x^{s+1}\,dx$

$= s\int\{\sum[x]-n+1)M_K(n)\}/x^{s+1}\,dx$

$= s\int \sum M_K(n)/x^s\,dx - s\int \sum nM_K(n)/x^{s+1} + p_K(s)$ where $p_K(s)$ is analytic for $\sigma > K+1$.

Thus $L_{(K+1)}(s) = \{(s/(s-1))L_K(s-1)\} + L_K(s-1) + p_K(s)$ and the result follows by induction.

We use this result to derive the main results about the oscillatory behaviour of the Möbius sum function and the higher summation functions.

Let $\underline{\sigma}$ be the smallest real number such that $\zeta(s) \neq 0$ for $\sigma > \underline{\sigma}$.

Theorem:

$\mathbf{M}_K(x) = \Omega_{+,-}(x^{K-1+\underline{\sigma}-\epsilon})$ as $x \to \infty$.

Proof

Suppose $M_K(n) + An^{(K-1+\underline{\sigma}-\epsilon)}$ is eventually of one sign.

Then the function defined by the Dirichlet series, $H_K(s) = \sum(M_K(n) + An^{(K-1+\underline{\sigma}-\epsilon)})/n^s$ has a singularity at the real point on its line of convergence.
It follows from the preceding theorem that
$H_K(s) = 1/[(s-1)(s-2)\ldots(s-K)\zeta(s-K)] + \zeta(s-K+1-\underline{\sigma}+\epsilon) + E_K(s)$ where $E_K(s)$ is analytic for $\sigma > K$.

Moving from right to left along the real axis we find the first singularity at $s-K+1-\underline{\sigma}+\epsilon = 1$.
i.e $\sigma = K+\underline{\sigma}-\epsilon$.

Since $H_K(s)$ is then analytic for $\sigma > K+\underline{\sigma}-\epsilon$ it follows that $\zeta(s-K)$ is analytic for $\sigma > K+\underline{\sigma}-\epsilon$.
In other words $\zeta(s)$ is analytic for $\sigma > \underline{\sigma}-\epsilon$. This contradicts the choice of $\underline{\sigma}$.

Notes

From the point of view of Peano arithmetic, the weakest statement about the oscillatory behaviour of $M_1(x)$ is the $\Omega_{+-}(x^{1/2-\epsilon})$ which is best possible if RH is true.

If we were set the task of proving each of :-

$M_1(x) = \Omega_{+-}(x^{1/2-\epsilon})$, $M_2(x) = \Omega_{+-}(x^{3/2-\epsilon})$, $M_3(x) = \Omega_{+-}(x^{5/2-\epsilon})$ ….. as $x \to \infty$, we would note that the Ω result for K does not imply the Ω result for K+1. For oscillating series generally the Ω result for K+1 is **stronger** than the Ω result for K. Yet we cannot find a 'largest K' to get the weaker results because K is unbounded.
In terms of later discussion the Ω theorems are regarded as an unbounded number of essentially different propositions.

The propositions $M_K(x) = \Omega_{+-}(x^{K-(1-\underline{\sigma})-\epsilon})$ as $x \to \infty$, increase in strength in arithmetic with increasing $\underline{\sigma}$. The collection with $\underline{\sigma} = ½$ is in a sense the weakest collection.

References

P. B. Braun. Topics in Number theory. D.Phil Thesis. University of Waikato 1979.