The Riemann hypothesis is undecidable in arithmetic (iii)

by

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Notation and usage

N denotes a natural number and Δ a real number.

Estimates $O(N^{\Delta}), o(N^{\Delta}), \Omega(N^{\Delta}),\ \Omega_{\pm}(N^{\Delta})$ assume $N \to \infty$. Δ is called the order of the asymptotic estimate under consideration. When we are working in rational arithmetic we assume Δ is rational.
ε will mean an arbitrarily small real/rational number greater than zero.
$A(N) = O\left(N^{\frac{a}{b}}\right)$ with a, b positive natural numbers means $|A(N)|^{b} = O(N^{a})$.
μ denotes the Möbius function, λ the Liouville function, Λ the von Mangoldt function, ζ the Riemann zeta function and ln denotes the natural logarithm.
Let s=σ+it in the usual complex variable notation with σ and t real numbers.

$$M(N) = \sum_{n\le N} \mu(n),\ \ g(N) = \sum_{n\le N} \frac{\mu(n)}{n}, \qquad S(N) = \sum_{n\le N} \lambda(n),\ \varphi(N) = \sum_{n\le N} \Lambda(n)\ , \qquad l(N) = \sum_{n\le N} \frac{1}{n} \text{ and}$$

$\pi(N)$ counts the number of primes bounded by N.

$$\zeta(s) = \sum_{n\ge 1} \frac{1}{n^{s}} \quad (\sigma > 1), \qquad \zeta(s) = (1 - \frac{2}{2^{s}})^{-1} \sum_{n\ge 1} \frac{(-1)^{n+1}}{n^{s}} \quad (\sigma > 0, s \ne 1\ ).$$

Let $f(s) = \sum_{n\ge 1} \frac{a(n)}{n^{s}}$ and $A(N) = \sum_{n\le N} a(n)$, then by definition $\mathrm{Cosum}_{N}\big(f(s)\big) = A(N)$.

We focus on finite rational arithmetic because we wish to isolate the arithmetic addition and multiplication done by computers in research work calculating the position of zeros of ζ in the critical strip. We see the count of a zero of ζ as essentially a sensible exercise in rational arithmetic where we are interpreting results from the 'perfect' theory in analysis.
The terms rational arithmetic, finite arithmetic, inductive arithmetic and Peano arithmetic will all mean arithmetic as discussed immediately below.

We simply mean Peano arithmetic free from the use of irrational numbers. For example, convergent series would only be used if they are known to have rational limit values. For example, the power series associated with the ratio of two polynomials with rational coefficients.
We have a practical orientation here. We may have a proof that A(N) =(CS)N+o(N), where CS denotes a valid Cauchy sequence, and then clearly we could still be in arithmetic as the notion of a Cauchy sequence is inductively based on rational sequences. However, if (CS) is a known irrational number it decreases the weight of what has been proved in the sense that the value of (CS) in arithmetic can only ever be approximated, and increases the weight of what has been proved in the

sense of using new constructions outside of arithmetic. We have the tension between the inward journey of new constructions in arithmetic and the outward journey of doing the new arithmetic using the new constructions. Thus if we know that (CS) is irrational from excursions into analysis we see this as an obstacle which cannot be worked around in rational arithmetic. Indeed, if we suppose an abstract proof of the result, we only have the two closed field operations on rational entities used finitely. We cannot possibly get the exact value of the first term in the asymptotic estimate in order to prove the error is as small as o(N). Our overall system of prescribed complex analysis includes only the types of proof developed using defined functions and constants and the pointwise manipulation of equations of the form

$X_{1,1}X_{1,2} \dots X_{1,k_1} + X_{2,1}X_{2,2} \dots X_{2,k_2} + \cdots \dots + X_{l,1}X_{l,2} \dots X_{l,k_l} = 0$ with the $X_{i,j}$ taking real values.

We wrap each and every X term in brackets like CS(X) to remind us that each term X is defined in arithmetic via a (C)auchy (S)equence.

We cannot however take a finite number of equations like this and obtain the precise value of an irrational X. The values of irrationals are out of reach of finite arithmetic.

However, since Cauchy sequences are defined inductively in arithmetic, we are able to follow the algebraic manipulation in prescribed analysis to levels of approximation.
Named functions and constants then become building blocks for more inter-relationships with the above fundamental structure. In arithmetic, evidence of the truth of the functional equation may be tested at any points we choose but it takes analysis to establish the precise truth within the rules and assumptions. The functional equation for ζ is an example of something which can be derived only in prescribed analysis. From arithmetic we would confidently expect point wise evidence consistent with the exact function equation for s≠0.

Introduction

We note from the theory of the Riemann zeta function that the actual order of M(N) is directly related to a vertical line in the complex plane splitting the plane into two regions, one of which is free from zeros of ζ and one which is not free from zeros. It is not decided at this point whether the line has zeros on it.
We may define this line in analysis from the relationship
$\sigma = \Theta = \text{lub}\{\Delta: M(N) = O(N^{\Delta})\} = \text{lub}\{\Delta: \zeta(s) \neq 0 \text{ for } \sigma > \Delta\}$.

We need to note there are propositions in arithmetic which imply the Riemann hypothesis (RH). For example, $M(N) = O(\sqrt{N})$ implies RH.
In terms of usage here we transport such an arithmetic proposition into the domain of real and complex analysis to establish the result. In itself, very little of the Riemann zeta function (if any) is recognizable in the domain of arithmetic but we are able to track the numerical integrity of the exact value results in numerical approximation via all the arguments used in establishing the theory. If for example we had a fully arithmetic proof that $M(N) = O(N^{\frac{1}{2}+\varepsilon})$ or $M(N) \neq O(N^{\frac{1}{2}+\varepsilon})$, we would call this a proof of RH or not(RH) in arithmetic, respectively, and it would make the conjecture decidable .

Imagine a mathematician working in arithmetic, shielded by a 'cone of silence' from analytical knowledge, handing a correct proof of one or the other to an analyst who then confirms that RH is either true or false depending on the arithmetic proof provided.

On the other hand, if the arithmetician hands the analyst a correct proof that the asymptotic order of M(N) in un-provable in arithmetic in the order range [0,1), the analyst may need to think about the ramifications for RH. This doesn't hand the analyst an obvious causal type proof of RH but we will show that it does imply that all calculated zeros of ζ in the critical strip will be simple zeros on $\sigma = 1/2$. Like the very simple formula for the sum of the first N numbers we get out somethings, when calculating zeros, which are implied from the assumptions in the domains we are working in. Thus we will use an inductive argument in arithmetic to conclude that calculated zeros lie on $\sigma=1/2$. This is a result about calculations, not about the singularities of $1/\zeta$ in $\sigma>0$.

We make up analysis from arithmetic and arithmetic must be the guardian of any outcomes. At the weak end of arithmetic we will find no numerical contradictions with results from prescribed analysis and at the strong end of arithmetic we have the finite results of finite induction which are always about exact values within arithmetic and such results have the highest level of certainty we find in proof in these domains.

By un-decidability of a proposed theorem in a domain of argument we mean that to decide whether the theorem is true, false or unprovable, we necessarily have to include assumptions outside the original domain to have any chance of resolving the problem. We require indisputable evidence, within our tolerance levels, that such a maneuver is truly necessary in the context.

The notion of set we use here is only for reference – indicating one, or a greater number, of characteristics required of the entities under discussion. For example, z is an element of the set C if and only if it is a complex number. C is just a reference in this context. We ignore any notion of the algebra of infinite sets.

One part of acceptable proof in this context is about a structured causal argument –argument as defined in the algebra of propositions which are either true or false. At the conclusion of the argument there is an object – a true or false theorem –and it doesn't require any further activity from anybody after any errors and omissions have been sorted out. On the other hand, there are many problems which are about some things existing or not existing or perhaps being spread between these two states in some patterned way. In some of these problems the arithmetician does not seem to be able to make progress in a particular domain.

An example is seen in the Möbius function, μ. We have a simple definition that $\mu(n) = 0$ if n is divisible by a square, otherwise (-1) if n is a product of r primes where r is odd and +1 if n is a product of r primes where n is even. This curious definition aligns the analytical relationship between the Dirichlet series in

$$\sum_{n\geq 1}\frac{\mu(n)}{n^s}\sum_{n\geq 1}\frac{1}{n^s} = \sum_{n\geq 1}\frac{\mu(n)}{n^s}\zeta(s) = \ 1 \ \{\sigma > 1\} \text{ because } \sum_{g|n}\mu(g) = 1\ (n=1), 0\ (n>1).$$

It is here that we find the connection between the Riemann hypothesis and the asymptotic order of the growth rate of M(N) in the classical analytic theory of the Riemann zeta function.

With knowledge of the theory of the Riemann zeta function in the domain of real and complex analysis we have

$M(N) = O(\sqrt{N})$ implies the Riemann hypothesis with all zeros simple (Ollyzko & te Riele [1]),

and, as a matter of fact,

$M(N) = \Omega(\sqrt{N})$ unconditionally (Titchmarsh [1] pages $371 - 372$).

Briefly, the first condition ensures that the μ Dirichlet series above is convergent for σ >1/2 and zeros of ζ are simple. The theorem below this follows because ζ has zeros on σ=1/2 and this defines the least upper bound of number σ which provide a half plane of convergence for the μ Dirichlet series. Any bigger half plane of convergence would prevent ζ having zeros on σ =1/2. This guarantees we have $M(N) = \Omega(N^q)$ for q<1/2 and the finer result for q=1/2 argument is completed in Titchmarsh [1] pages 371-372.

However, if we wish to contemplate these two pieces of information in rational arithmetic we have quite a different story.

We prove via Theorem 1 below that the true statement in analysis that M(N)=o(N) is unprovable in arithmetic. We go on to show this ensures that all calculated zeros of ζ in the critical strip will be on σ =1/2 and will be simple zeros.

Theorem 1

Let a(n) ∈ {-1,1 0} be any defined/specified arithmetic sequence satisfying

$$\sum_{n\le N} |a(n)| = \kappa N + o(N) \text{ and } \sum_{n\le N} a(n) = o(N),$$

where κ is a proven irrational. Then

$$\sum_{n\le N} a(n) = o(N) \text{ is unprovable in arithmetic.}$$

Proof:

Let

$$A(N) = \sum_{n\le N} a(n) \text{ and suppose we have proved in arithmetic that } A(N) = o(N).$$

Let

$$A^+(N) = \sum_{\substack{n\le N\\ a(n)>0}} 1 \text{ and } A^-(N) = \sum_{\substack{n\le N\\ a(n)<0}} 1 .$$

Then

$$A^+(N) - A^-(N) = o(N) \text{ and } A^+(N) + A^-(N) = \kappa N + o(N).$$

Consequently

$$2A^+(N) = \kappa N + o(N) \text{ and hence } \frac{2A^+(N)}{N} = \kappa + o(1), \qquad \text{and}$$

$$2A^-(N) = \kappa N + o(N) \text{ and hence } \frac{2A^-(N)}{N} = \kappa + o(1).$$

We thus would have proved in rational arithmetic that the rational sequences

$\left\{\frac{2A^+(n)}{n}\right\}$ and $\left\{\frac{2A^-(n)}{n}\right\}$ converge to a number which is not rational.

This cannot be done in arithmetic. The irrationals are persona non-grata in this domain.

Consider the estimates for rational Δ,

$$A(N) = O(N^{\Delta})\ \Delta > 1, \quad A(N) = O(N^{\Delta})\ \ \Delta < 0, \quad A(N) = O(N^{1}), \quad A(N) = O(N^{\Delta})\ 0 \le \Delta < 1,$$

The first, second and third are provable and the fourth estimate is unprovable in arithmetic. Theorem 1 establishes we cannot exhibit an example in the last range satisfying the conditions of the theorem because it would involve establishing the exact value of an irrational number. Hence

Corollary 1

In arithmetic we are free to choose any rational value we like for Δ in the range $[0,1)$ and we will not be able to contradict this in arithmetic.

Application to the Riemann hypothesis

It is well established in the literature from both real and complex analysis that

$$M(N) = \sum_{n\le N} \mu(n) = o(N) \text{ and } g(N) = \sum_{n \le N} \frac{\mu(n)}{n} = o(1).$$

The classical theory of ζ provides the result

$$\varphi(N) - N = \mathrm{Cosum}_N\left(\frac{\zeta'(s)}{\zeta(s)} + \zeta(s)\right) = o(N), \text{using contour integration and using a zero free region}$$

for ζ in the critical strip. (See for example Tenenbaum [1] pages 167-168).

Using the same method and the relationship $\displaystyle\sum_{n\ge 1} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$ we are able to prove

$$\sum_{n\le N}|\mu(n)| - \frac{6}{\pi^2}N = \mathrm{Cosum}_N\left(\frac{\zeta(s)}{\zeta(2s)} - \frac{6}{\pi^2}\zeta(s)\right) = o(N).$$

i.e.

Lemma 1

$$\sum_{n\le N}|\mu(n)| = \frac{6}{\pi^2}N + o(N).$$

See for example (Insuk K. and Cho M.H. [1]).

Consequence for the Möbius sum function

It is very well known that $6/\pi^2$ is irrational and so applying Theorem 1 we see that M(N)=o(N) is unprovable in arithmetic.

Also, from the equation $M(N) = -\sum_{n\le N} ng(n) + Ng(N)$, we see $g(N) = o(1)$ implies $M(N) = o(N)$

hence $g(N) = o(1)$ is also unprovable in arithmetic.

Notes:

A local argument, more clearly prime number based, confirming the validity of these results, is summarized in Appendix (1). (See also Braun [1]).

We thus see that although the sequence $\{\mu(n)\}$ is inductive in that it can be evaluated stepwise for n=1,2,3 … it is essentially non-inductive from the consequences of Lemma 1. There is no arithmetical pattern in the sequence which allows a first term asymptotic expression for the sum of the absolute values of the first N terms. This would appear to be a sensible definition for a non-inductive sequence.

Consequences for the Riemann hypothesis

Here, we explain why calculated zeros of ζ lie on σ =1/2 and are simple zeros.

Keep in mind with the big O order of M(N) unprovable in the range [0,1) in arithmetic, that in two valued logic, we only have null or empty propositions if we attempt to use them to derive analytical properties in the theory of ζ. We note the logical implication ‘ $M(N)=O(N^a)$ implies $M(N)=O(N^b)$ if a<b’ does not have meaning in arithmetic because the statement is essentially empty or null. The two components about the order of M(N) are neither true nor false. At best they are undecidable.

ζ is initially defined in Re{s}>1 so we think about an arithmetic/analytic continuation into Re{s}>0. For the choice q we'll use μ_q to signal an assumption $M(N)=O(N^q)$. From the equation

$\sum_{n\geq1}\frac{\mu_q(n)}{n^s}\sum_{n\geq1}\frac{1}{n^s}=1$ we get the analytic continuation of $\frac{1}{\zeta}$ into $\mathrm{Re}\{s\}>q$ with no singularities.

Since this is unresolvable in arithmetic, we have to look to the analytic theory to establish where the zeros of ζ are located in the critical strip. In arithmetic we will be able to say –yes or no if the analyst proves there is a zero in a certain region. There is a systematic way of counting the zeros of ζ(σ+it) within the analytical rules and assumptions using arithmetic in computation, even though arithmetic cannot verify the exact values of zeros numerically. In a sense, the absurd proof that RH is true because ζ does not have any zeros in the critical strip just needs the condition (exact values of zeros unprovable in arithmetic) to give it a non-zero amount of sense. The existence of zeros is actually determined in analysis indirectly where we prove under certain conditions on a real valued function then if it changes sign on an interval [a,b] then there is a point in the interval where the value of the function is exactly zero. More sophisticated theorems are needed in complex analysis to count multiple zeros. These are not theorems in arithmetic.

Now we do have some solid analytical results, the most important being that ζ has a zero on σ =1/2 and in fact quite a few – but we only need one.

Thus, in the analytical world the order of M(N) is necessarily greater than or equal to 1/2 and we would expect this analytical truth to be reflected in numerical investigation. Although we can only investigate very small samples of ordered natural numbers and their μ values, the early evidence shows the expected consistency between arithmetic and prescribed analysis.

With the reported existence of a zero in σ >1/2 the arithmetician has to face the analytical truth that the order of M(N) is necessarily greater than or equal to Δ for some Δ>1/2.
We then have two impositions on arithmetic: one that Θ=1/2 and the other that Θ>1/2 and the arithmetician has no way of rejecting one over the other. It is a dilemma the arithmetician has created and recognises. Numerical investigation into zeros has to reflect this and thus in arithmetic the existence of a zero in σ >1/2 cannot be indicated by computation because it would resolve which of the two arithmetics was in fact a ghost imposition.
Similarly, we know from analysis [1] Ollyzko A.M. & te Riele H.J.J. that M(√N) implies zeros in the critical strip are simple zeros. Since this estimate is unprovable in arithmetic we will not be able to see this contradicted in numerical calculation. We record in Note 2 below that from an analytic point of view in the abstract, the actual best possible big O estimate does not need to be O(√N) as for example O(√N ln(N)) produces the same best possible order estimate.

Note 1

It is interesting to compare the Theorem 1 result applied to μ with a result about the Liouville function. In this case we have

$$\sum_{n\leq N} |\lambda(n)| = 1N + 0$$

which is a very sharp asymptotic estimate but the coefficient of N is rational and we cannot apply Theorem 1.

However, since

$$M(N) = \sum_{n\leq N} \mu(n) S(\frac{N}{n^2})$$

we see that if S(N)=o(N) is provable in arithmetic then so is M(N) and hence S(N)=o(N) is unprovable in arithmetic.

The 'irrationality' factor for S(N) shows up in a more subtle way:- we may define the Liouville function from the equations

$\sum_{n\leq N} \lambda(n)[\frac{N}{n}] = [\sqrt{N}]$ $(N \geq 1)$ where [ ] stands for the greatest integer function.

Although $[\sqrt{N}]$ always has interpretation in arithmetic, there is difficulty expressing this interpretation in the limit case and the o estimate for S(N) is about the limit case.

Note 2

Although it is almost more than fashionable to have the natural numbers seamlessly imbedded in the real and complex fields it is important to realise that Θ=1/2 in analysis is essentially a pragmatic imposition on arithmetic where we simply cannot get agreement on the Ω and the O best estimates and the calculations of zeros must reflect this. Although $M(N) = O(\sqrt{N})$ cannot be contradicted in arithmetic we see for example in [1] Odlyzko & te Riele that it is a sufficient condition for simple zeros and may not be necessary. It may transpire in analytic work that $M(N)=\sqrt{N}\omega(N)$ where $|\omega(N)|$ is unbounded.

Where this sort of phenomenon has been established in a related question we have

$$\varphi(N) = \sum_{n\leq N} \Lambda(N) = N + E(N) \text{ where } E(N) = \Omega(\sqrt{N}\ln(\ln(\ln(N)))),$$

(Titchmarsh [1] page 374). Here though we are not dealing with a function defined in arithmetic, as φ(N) uses the values of ln(p) for p prime.

Note 3

Theorem 1 and the application show that we have an underlying negative cause for calculated zeros of ζ lying only on σ =1/2 in the critical strip without their being too much about prime numbers involved. This, in the author's view, is different from a positive causal proof that ALL zeros in the critical strip lie on σ =1/2. The above argument says we will not find a contradiction by calculation but we can only ever do a finite number of calculations and hence is does not amount to an argument which holds in the limit.

We give a brief argument which may appear as a 'tongue in cheek' argument that RH doesn't have anything to do with prime numbers but it does highlight the 'limit' type nature of the problem outside of the primes in arithmetic.

Let $\zeta_N(s) = \prod_{p_k \geq p_N} (1 - \frac{1}{p_r^s})^{-1}$ where the primes are ordered $p_1 < p_2 < \cdots .. < p_k \ldots$.

Then for σ>1

$$\ln(\zeta_K(s)) - \ln(\zeta(s)) = \sum_{i<N} \frac{1}{p_i^s} + \frac{1}{2}\sum_{i<N} \frac{1}{p_i^{2s}} + \frac{1}{3}\sum_{i<N} \frac{1}{p_i^{3s}} + \ldots \ldots . = \vartheta\ (s) \text{ where } \vartheta \text{ is an entire function.}$$

In particular, this provides the analytic continuation of $\ln(\zeta_N(s))$ into σ>0 .

Hence in σ>0

$$\zeta_K(s) = \zeta(s)e^{\vartheta\ (s)}$$

and so $\zeta_K(s)$ has the same zeros and analytic profile as $\zeta(s)$ in $\sigma > 0$.

We may thus omit the prime numbers one by one without stopping and still be contemplating the analytic character and zeros of ζ in σ>0. By appealing to mathematical induction we conclude that the Riemann hypothesis has nothing to do with the values of the finite primes of arithmetic.

Likewise, a similar observation then may be harnessed in the question of the order of the growth rate of M(N) for example. The analytic profile of $\ln(1/\zeta(s))$ and $\ln(1/\zeta_k(s))$ match in Re{s}>0.

We may omit numbers involving the first k primes from the μ values and we still have the same problem as far as RH is concerned. This is true for k=1,2…….. .

This alone suggests that an arithmetic non- trivial order estimate M(N) is impossible in arithmetic.

Unfortunately, the 'for all' of arithmetic and the 'for all' of analysis do not coincide in the limit which gives an indication why there is sense in talking about arithmetic primes and analytic primes.

Note 4

Clearly, the type of argument in the main body of the discussion could apply to a function defined by

$$L(s,\chi) = \prod_{p\ \text{prime}} (1 - \frac{\chi(p)}{p^s})^{-1}$$

and which satisfies the conditions of Theorem 1, with more relaxed conditions on the rational values of χ and a functional equation of a similar kind to the ζ functional equation.

Epilogue

The Riemann hypothesis is a problem in which we need to make a distinction between the analytic primes and the arithmetic primes. This is not particularly popular as the fashion is to imbed the rational field in the complex field and get on with things. We overlook too much by doing this in the case of the zeta function.
We really do need to remember that the domain defined by complex analysis requires more assumptions than mathematical activity in the rational field. We have noted above the 'for all' of rational arithmetic is different from the 'for all' of complex analysis. We can only ever obtain the exact values of irrational numbers in a theoretical sense but the ability to obtain accurate approximations in arithmetic, inherent in the construction of the reals, ensures that in theorems about values derived in prescribed complex variable we will not find numerical contradiction. Such a theorem will be either provable or unprovable in arithmetic.
This result then is one point at which analytic prime number theory actually imposes a 'truth' on arithmetic. In arithmetic we know we cannot contradict the analytic result numerically. In the terminology of abstract set theory and logic the problem is independent from the axioms of arithmetic and we are free to believe there may be a zero outside the confines of numerical verification. All we are able to say about this is that it would require assumptions outside of the axioms requires to construct develop prescribed analysis.

There is a difference between the two references,

$$\{\mu(n): n \geq 1\} \text{ and } \{\mu(n): n \geq 1 \text{ and } \sum_{n\geq 1} \frac{\mu(n)}{n} = 0\}$$

from the point of view of arithmetic.

We are able to make this distinction in some cases where the exact values of certain known irrationals appear in the theory as above.

The standard proof that $\sqrt{2}$ is irrational clearly requires the existence of $\sqrt{2}$ and where the two argument systems are separated there is no $\sqrt{2}$ to talk about. The theorem that no rational number squared equals 2 is a good theorem in rational arithmetic.
In Braun [2] we discuss ramifications of this in the context of the Lindelöf hypothesis.

Appendix 1

We summarize here a more specific argument leading to $M(N) = o(N)$ and $g(N)=o(1)$ as unprovable propositions in arithmetic. The detail leading to the conclusion is in Braun [1] .

Basically, the Möbius sum function M and the prime number counting function π are orderly scrambled up versions of each other in the sense that their values may be derived from each other in arithmetic through simple algebraic processes.

In the limiting case, the prime number counting function defines the exact values of the irrational sequence $\ln(p)/\ln(2)$ for p prime. We thus argue that certain limiting values of M used in asymptotic estimates include the precise values of these irrationals and such ability must be outside arithmetic.

i.e. $\{\mu(n): n\geq 1\}$ contain no information about the asymptotic order of $|M(N)|$ but the analytic

$$\left\{\mu(n): n \geq 1 \text{ and } \sum_{n\geq 1} \frac{\mu(n)}{n} = 0\right\}$$ contains the necessary information to define $\ln(p)/\ln(2)$

and is thus outside of rational arithmetic. Recall these 'sets' are for reference. Anyone glued to the notion that the contents of sets are independent of human thought may not be able to see a difference. i.e. mindless sets and intelligence sets.

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