Another irrational sums argument for the Riemann hypothesis*

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Let μ denote the Möbius function.

Let $M(x) = \sum_{n \leq x} \mu(n)$.

Let [x] = x-{x} where [ ] denotes the greater integer function and let {x} denote the fractional part of x in situations clearly involving the greatest integer function.

Let Θ =lub {Δ: ζ(s)≠0 for σ>Δ}.

It is well known that Θ is the precise order of M(x), and defines the precise half plane (σ> Θ) of

convergence of the Dirichlet series for $\frac{1}{\zeta(s)}$.

i.e. $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \zeta(s) = 1 \quad (\mathrm{Re}\{s\} = \sigma > \Theta)$.

Thus

$$\mathrm{RH} \equiv M(x) = |O\left(x^{\Theta}\right)$$

where the notation |O is used to indicate exact order of asymptotic growth as a power of x as x→∞.

Since |μ(n)|≤1 we also have

$$\mathrm{RH} \equiv M(2x^2) = |O\left(x^{2\Theta}\right) \ \ldots\ldots\ldots\ldots (1).$$

We may not be able to prove this in arithmetic (UD1) but we do know that we will not find a numerical contradiction to this in arithmetic.

Section 1

An irrational summation for $M(2x^2)$

Theorem

A non-trivial asymptotic estimate for $M(2x^2)$ in UD1 is impossible.

i.e. In UD1 we cannot argue an asymptotic estimate better than the trivial $M(2x^2) = O(x^2)$ as $x \to \infty$.

Proof

Line by line

$$\sum_{1\le n\le x} \mu(n)[\frac{x}{n}] = 1 \quad (x \ge 1)$$

$$\sum_{1\le n\le x} \sum_{g|n} \mu(g) = \sum_{1\le n\le x} \mu(n)[\frac{x}{n}] = 1 \quad (x \ge 1)$$

$$M(x) = \sum_{1\le m\le x} \mu(m) \sum_{1\le n\le \frac{x}{m}} \mu(n)[\frac{x}{m}].$$

Hence, for $x\geq 1$

$$M(x) = \sum_{1\le m\le x} \mu(m) \sum_{1\le n\le x} \mu(n)[\frac{x}{nm}].$$

Thus

$$M(x) = \sum_{1\le m\le x} \mu(m) \sum_{1\le n\le \sqrt{x}} \mu(n)[\frac{x}{nm}] + \sum_{1\le m\le x} \mu(m) \sum_{\sqrt{x}<n\le x} \mu(n)[\frac{x}{nm}].$$

$$= \sum_{1\le m\le \sqrt{x}} \mu(m) \sum_{1\le n\le \sqrt{x}} \mu(n)[\frac{x}{nm}] + \sum_{1\le m\le \sqrt{x}} \mu(m) \sum_{\sqrt{x}<n\le x} \mu(n)[\frac{x}{nm}] +$$

$$+ \sum_{\sqrt{x}<m\le x} \mu(m) \sum_{1\le n\le \sqrt{x}} \mu(n)[\frac{x}{nm}] + \sum_{\sqrt{x}<m\le x} \mu(m) \sum_{\sqrt{x}<n\le x} \mu(n)[\frac{x}{nm}].$$

The second and third terms are equal and the fourth term is zero.

Hence

$$M(x) = \sum_{1\le m\le \sqrt{x}} \mu(m) \sum_{1\le n\le \sqrt{x}} \mu(n)[\frac{x}{nm}] + 2 \sum_{1\le m\le \sqrt{x}} \mu(m) \sum_{\sqrt{x}<n\le x} \mu(n)[\frac{x}{nm}]$$

$$= \sum_{1\le m\le\sqrt{x}} \mu(m) \sum_{1\le n\le\sqrt{x}} \mu(n)[\frac{x}{nm}] + 2 \sum_{1\le m\le\sqrt{x}} \mu(m)\{ 1 - \sum_{1\le n\le\sqrt{x}} \mu(n)[\frac{x}{nm}] \}$$

$$= - \sum_{1\le m\le\sqrt{x}} \mu(m) \sum_{1\le n\le\sqrt{x}} \mu(n)[\frac{x}{nm}] + 2M(\sqrt{x}).$$

$$M(x) = \sum_{1\le m\le\sqrt{x}} (2\mu(m) - \sum_{1\le n\le\sqrt{x}} \mu(m)\mu(n)[\frac{x}{nm}]) \ldots\ldots\ldots (2).$$

Hence,

$$M(2x^2) = \sum_{1\le m\le\sqrt{2}x} (2\mu(m) - \sum_{1\le n\le\sqrt{2}x} \mu(m)\mu(n)[\frac{2x^2}{nm}]) \ldots\ldots.. (3).$$

i.e. $M(2x^2) = \mathcal{F}\left((\sqrt{2})x\right)$ in a non trivial way in arithmetic (UD1).

We could note that $M(2x^2) = M(\left(((\sqrt{2})x)^2\right)$ but this is a relationship in UD2.

Although we have been working in UD2 to get the RHS of (3) we note that in this context $1 \le n \le (\sqrt{2})x$ is summation over n satisfying $n^2 \le 2x^2$ and this is a valid summation in UD1.

We could interpret $\sqrt{2}x$ is a shorthand way of defining the extent of the summation- $\{n: n^2 \le 2x^2\}$ which is something which may be calculated in arithmetic for $x = 1,2,3, \ldots..$.

However, this summation cannot be reduced to a finite number of calculations in UD1 since such an inductive mechanism would provide a finite arithmetic evaluation of $\sqrt{2}$.

The summation $\sum_{1\le m\le\sqrt{2}x} (\mu(m) + \sum_{1\le n\le\sqrt{2}x} \mu(m)\mu(n)[\frac{2x^2}{nm}])$ may look innocent enough but has

imbedded in it an ever increasing amount of distinct logical information as $x\to\infty$, and it is this we necessarily need to take into account in UD1 in order to derive a non-trivial asymptotic estimate for $M(2x^2)$.

i.e. as $x\to\infty$ we need to take into account an unbounded amount of logically distinct information in UD1 to show that cancellation in the terms of the summation reduces the asymptotic estimate below the trivial $O(x^2)$ as $x \to \infty$.

It is only the irrationals in UD2 which in their construction contain unbounded information.

Likewise, the trivial estimate cannot be established as best possible in UD1 because this implies that non- trivial estimates can be proven false.

Thus a non trivial estimate for $M(2x^2)$ in UD1 is impossible.

It is the derivation of a summation of the form $\mathcal{F}\left((\sqrt{2})\mathrm{x}\right)$ which exposes the irrational nature of the summation in UD1.

Thus RH is unprovable in UD1.

Corollary

Zeros of ζ in the critical strip indicated by numerical calculation will lie on σ=1/2 and be simple.

Proof

A zero off the line σ=1/2 would give a numerical contradiction to the un-decidability result and a repeated zero would contradict the UD1 un-decidable proposition $\mathrm{M(x)=O}(\sqrt{x})$.

Section 2

Unprovability of the prime number theorem in arithmetic (UD1)

Let $\mathrm{l(x)} = \sum_{1\le n\le x} \frac{1}{\mathrm{n}}$ and $\pi(\mathrm{x}) = \sum_{\substack{\mathrm{p\ prime} \\ \mathrm{p}\le\mathrm{x}}} 1.$

Corollary:

The prime number theorem in the form $\pi(\mathrm{x}) \sim \frac{\mathrm{x}}{\mathrm{l(x)}}$ is unprovable in UD1.

Proof

The prime number theorem is logically equivalent to $\mathrm{M(x) = o(x)}$ as $\mathrm{x} \to \infty$ (Chandrasekharan [1], Gelfond and Linnik [1]]) and this a non-trivial estimate for M(x), which as discussed above, cannot be argued inductively in UD1.

We note that elementary proofs of the prime number theorem make extensive use of the natural logarithm function and this is a function necessarily in UD2.

Notes

There is a simple interpretation of (2) in terms of Dirichlet series products:

The coefficient sum of the Dirichlet series product $\mathrm{L}^2_{\sqrt{\mathrm{x}}}(\mathrm{s})\zeta(\mathrm{s}) = (\sum_{1\le n\le\sqrt{x}} \frac{\mu(\mathrm{n})}{\mathrm{n^s}})^2\zeta(\mathrm{s})$

is $\sum_{1\le n\le x}\sum_{1\le m\le x} \mu(\mathrm{n})\mu(\mathrm{m})\left[\frac{\mathrm{x}}{\mathrm{nm}}\right].$

The coefficient sum of $L_{\sqrt{x}}^2(s)\zeta^2(s)$ is unity for numbers up to x and hence from

$(1 - L_{\sqrt{x}}(s)\zeta(s))^2 = 1 - 2L_{\sqrt{x}}(s)\zeta(s) + L_{\sqrt{x}}^2(s)\zeta^2(s)$ we see the coefficient sum of

$\frac{1}{\zeta(s)} - 2L_{\sqrt{x}}(s) + L_{\sqrt{x}}^2(s)\zeta(s)$ is zero for numbers up to x. Equation (2) now follows.

References

[1] Chandrasekharan K. Arithmetical Functions page 25. Springer-Verlag. 1970.

[1] Gelfond A.O. and Linnik Yu.V. Elementary methods in Analytic Number Theory. Rand McNally & Co. 1965.

*See 'Lindelöf hypothesis revisited' for irrational sums and 'The Euler-Mascheroni constant and the Riemann hypothesis' for discussion of UD1 and UD2.
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