Is the Lindelöf Hypothesis undecidable? – Peter Braun*

The Lindelöf hypothesis that $\zeta(s) = O(t^{\epsilon})$ as $t \to\infty$ for $\sigma \geq ½$ is another conjecture in number theory with the property that work towards the conjecture is seemingly converging on the result but it still remains resistant to complete proof.

We know that the Lindelöf hypothesis is equivalent to the following statement about the zeros of the zeta function: for every $\varepsilon > 0$, the number of zeros with real part at least $1/2 + \varepsilon$ and imaginary part between T and T + 1 is o(log(T)) as T tends to infinity. The Riemann hypothesis implies that there are no zeros at all in this region and so implies the Lindelöf hypothesis. The number of zeros with imaginary part between T and T + 1 is known to be O(log(T)), so the Lindelöf hypothesis seems only slightly stronger than what has already been proved.

However, we also know that if M(x) denotes the Möbius sum function, that $M(x) = O(x)$ as $x\to\infty$ is trivial but $M(x) = o(x)$ as $x \to\infty$ is equivalent to the prime number theorem, Gelfond and Linnik [2]. This is far from a simple upgrade in a hierachy of argument.

A general result in Titchmarsh [3] may assist in showing that the Lindelöf hypothesis is undecidable in much the same way that the Riemann hypothesis has been discussed as being undecidable, Braun [1]. As the Riemann hypothesis implies the Lindelöf hypothesis this would then be sufficient grounds to secure the unprovability of the Riemann hypothesis.

The result in question is that a Dirichlet series f(s) is convergent in the half plane where f(s) is regular and $\mu(\sigma) = 0$. The function $\mu(\sigma)$ is the convex downward function discussed in Titchmarsh [3] relating to functions of finite order.

If we then consider the functions $f_K(s) = \{(1-2/2^s)\,\zeta(s)\}^K$, K= 1,2, 3.... we see that these conditionally convergent Dirichlet series are all convergent for $\sigma > ½$ assuming the Lindelöf hypothesis.

On the face of it these propositions for K = 1, 2, 3 ... are essentially logically different in the sense that we may be able to construct conditionally convergent series for which this property is not true.

If this is the case we would be trying to fit an unbounded number of different propositions into a finite universe of argument and the undecidability of the Lindelöf hypothesis follows.

* See also: Lindelöf hypothesis revisited (13 Mar 2016) www.peterbraun.com.au

References

[1] P.B. Braun. A note on the Riemann hypothesis (Draft 2). www.peterbraun.com.au

[2] A.O.Gelfond & Yu.V.Linnik. Translation. Elementary methods in Analytic Number Theory. Rand McNally and Company, Chicago, 1965

[3] E. C. Titchmarsh. The Theory of Functions. Oxford University Press, 2nd edition onward, 1979.