**Background comment and thesis notes**

The link below this link provide scanned copies of the first three sections of a D. Phil thesis –

Topics in Number Theory – P. B. Braun. (University of Waikato 1979).

The notes were scanned without correcting spelling or other typographical errors. We note that Titchmarsh should replace Titchmarch throughout.

The notes cover the methods used to derive the equivalences used in the earlier note on the Riemann hypothesis. The oscillatory behaviour of sum functions is also discussed.

The Möbius sum function shows up in the coefficients of the Dirichlet series representation for $1/\zeta(s)$ and the corresponding sum function and the higher sums all have oscillatory behaviour related to the truth of the quasi Riemann hypothesis:- $\zeta(s) \neq 0$ for $\sigma > \alpha$.

Various results about oscillatory behaviour (or lack of it) have appeared in the literature over the years but a systematic study does not seem to be readily available.

It is hoped that the following notes give a straight forward indication of how logical equivalences may be developed in this area of study.

The oscillatory behaviour comes into play using the observation about Dirichlet series with real coefficients that if the coefficients are eventually of one sign then the function represented by the series has a singularity at the real end point of convergence of the series.

The higher sums are formed by summing the sums.

That is,

$M_1(x) = \sum\mu(r)$ where $\mu$ is the Möbius sum function, and

$M_n(x) = \sum M_{(n-1)}(n)$

where in each case summation is assumed to be to [x].

We are able to form an unbounded sequence of statements each of which is logically equivalence to the Riemann hypothesis.

Namely

$$
\begin{aligned}
[M_1(x) = O(x^{(1/2)+\varepsilon}) \text{ as} \to\infty] &\equiv [M_2(x) = O(x^{(3/2)+\varepsilon}) \text{ as} \to \infty] \\
&\equiv [M_3(x) = O(x^{(5/2)+\varepsilon}) \text{ as} \to \infty] \\
&\equiv [M_4(x) = O(x^{(7/2)+\varepsilon}) \text{ as} \to \infty] \\
&\equiv [M_5(x) = O(x^{(9/2)+\varepsilon}) \text{ as} \to \infty]
\end{aligned}
$$

and so on.

The stated estimate for $M_n(x)$ always follows from the $M_{(n-1)}(x)$ estimate without qualification.

Implication in the reverse direction does not generally follow for any pair and counter examples are easily constructed.

Indeed, the best we can hope for generally with

$A_k(x) = O(x^d)$ as $x \to \infty$ is $A_{(k-1)}(x) = O(x^d)$ $x \to \infty$ .

For example let $a(n) = (-1)^{n+1}n$.

Then $A_1(x)$ and $A_2(x)$ have the same order and these may also be used to define higher sums $B_n(x)$ and $B_{(n+1)}(x)$.

Let $P_n$ be the proposition $[M_n(x) = O(x^{(n-(1/2)+\varepsilon)})$ as $\to \infty]$.

Then $P_1, P_2, P_3, \ldots P_n, \ldots$ are ever weakening in this clearly definable way.

The extent to which unbounded chains of logically equivalent propositions have been studied is not known to the writer.

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